Issue 12

April 2007

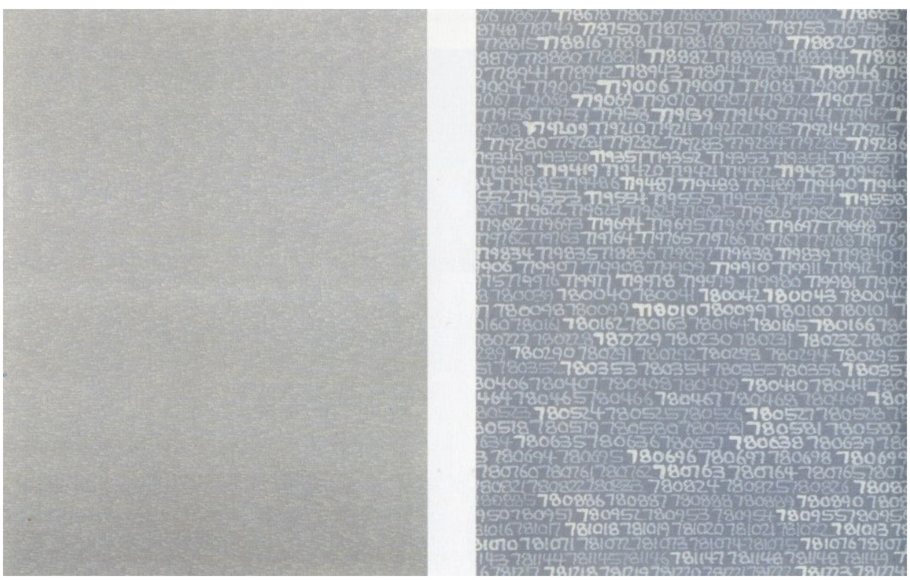
[a r k h a ï]

Αρχαι

www.arkhai.com

exploring infinity: number sequences in modern art

Götz Pfander and Isabel Wünsche



1. 2 is a prime number so put a circle round the 2 on your square.

1	2	3	~~4~~	5	~~6~~
11	~~12~~	13	~~14~~	15	~~16~~
21	~~22~~	23	~~24~~	25	~~26~~

2. Now cross out all the other multiples of 2

If you can spot a pattern you will be able to do this very quickly.

3. Now put a circle round the next number after 2 that isn't crossed out. ie. 3

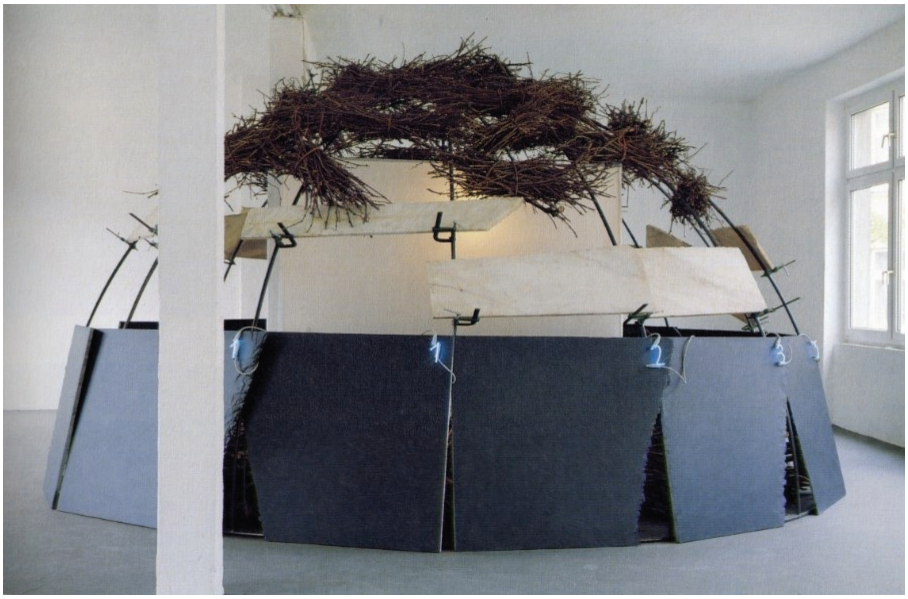
1	2	3	~~4~~	5	~~6~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~

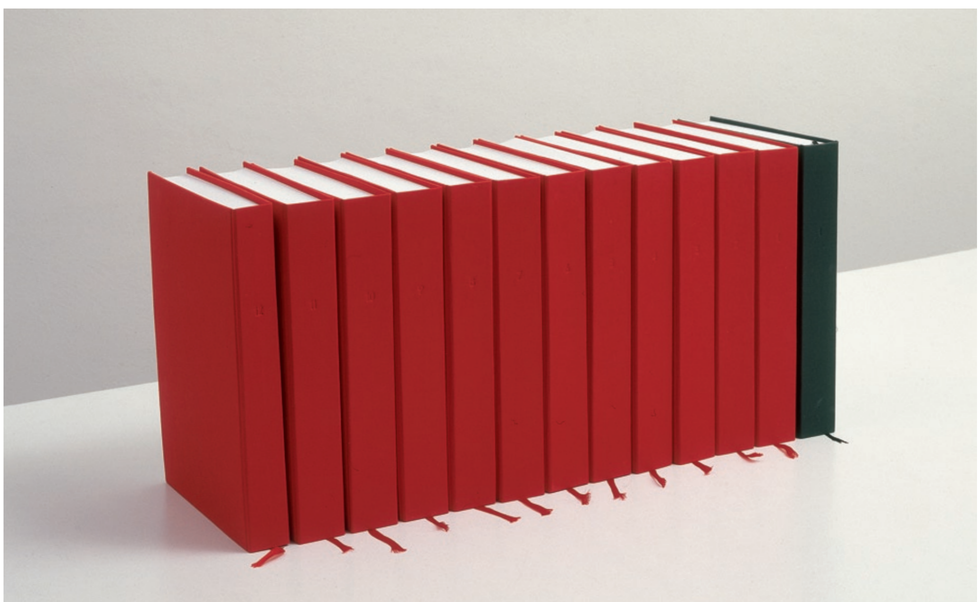
4. Now cross out all the other multiples of 3

You will find that some of them are crossed out already.

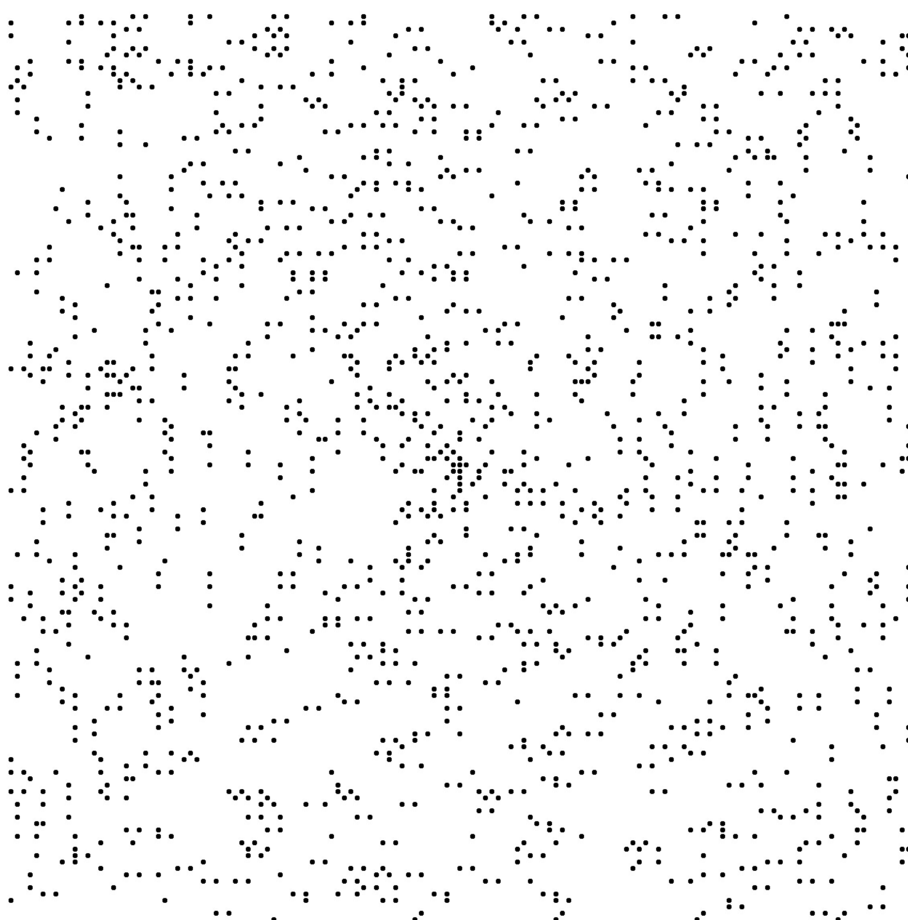
Once again, look for a pattern.

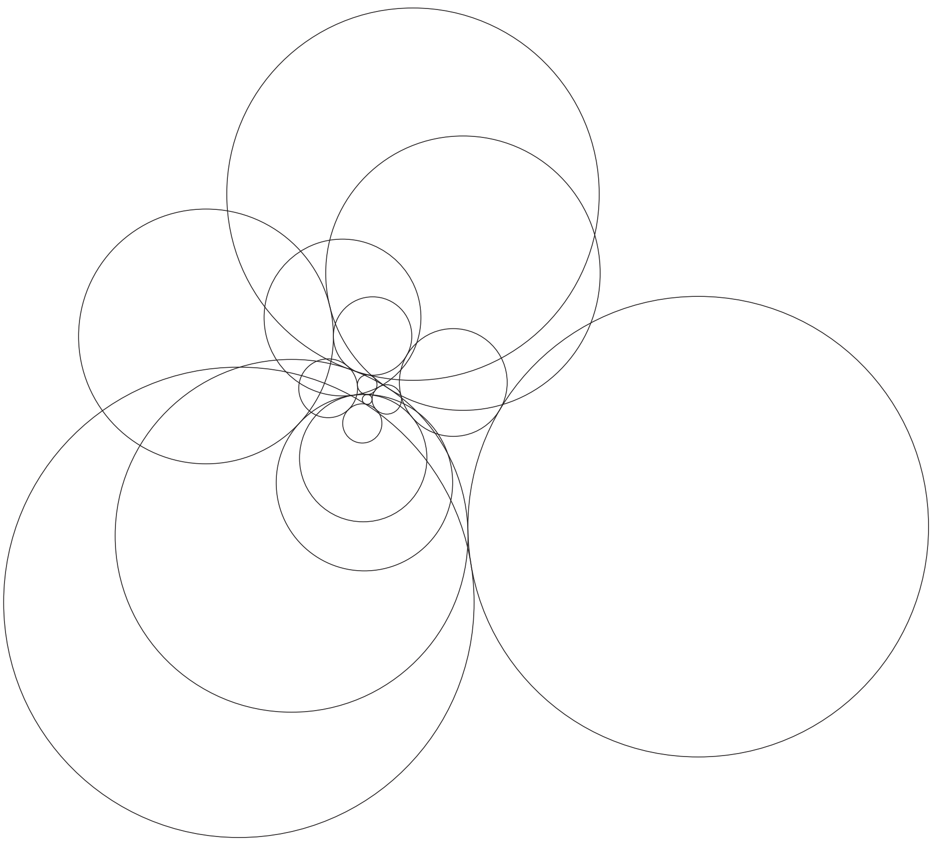
5. Now move on to the 5. Circle it and cross out all its other multiples. Keep going.....all the circled numbers will be prime!!






47
Teil 1. Vermutung des letzten Paares
48
Teil 2. Vermutung des letzten Paares





Illustrations:

page 42

Roman Opalka, *OPALKA 1965/ 1-∞,* 1965, Detail 774 509 – 800 148
Image from: Karin von Maur, *Magie der Zahl in der Kunst des 20. Jahrhunderts*, Stuttgart, 1997, Ill. 7.47 and 7.47a, p. 175.

page 43

Rune Mields, *Das Sieb des Eratosthenes III* (The Sieve of Eratosthenes III), 1977
Image from: http://home.ph-freiburg.de/grevsmuehlfr/material/forschung/2-2%20F%E4cherverbindende%20Studien/MT%20124%20-%20NUMBER%20INVESTIGATIONS.pdf

page 44

Mario Merz, *Pythagoras' Haus*, 1994
Image from: Karin von Maur, *Magie der Zahl in der Kunst des 20. Jahrhunderts*, Stuttgart, 1997, Ill. 9.4, p. 205.

page 45

Michael Müller, *Ulam Rot/Grün* (Ulam Red/Green), 2005
Image from the artist.

page 46

Michael Müller, *47 48*, 2005
Image from the artist.

page 47

Michael Müller, *Ulam Spirale* (Ulam Spiral), 2006
Image from the artist.

page 48

Michael Müller, *Ulam Ringe* (Ulam Rings), 2005
Image from the artist.

page 49

Michael Müller, *Ulam Gebirgszug* (Ulam Mountain Range), 2005
Image from the artist.

A fascination with numbers has shaped the work of numerous artists throughout the centuries. Whereas in earlier times artists focused on the mystical symbolism of individual numbers and their perceived capacity to give structure to the complexity of the universe, modern artists have concentrated more on an analysis of the pictorial language and the conceptual dimension of numbers. Their artistic approaches vary greatly in respect to both form and content and often counter the rational thought structures of our age with alternative ways of thinking and seeing. Numbers and numerical systems play an important role in the work of artists such as Hanne Darboven, Charles Demuth, Jasper Johns, Joseph Kossuth, Mario Merz, Rune Mields, Michael Müller, On Kawara, Roman Opalka, and others [1]. As Karin Maur pointed out in a recent exhibtion catalogue, these artists use numbers as signs of modernity, found objects and random subjects. They explore numerical writing and the relationships between numbers and languages, and investigate the development from image to arithmetical progression, codes of war, and alogical calculation. They reflect upon the role of numbers in nature, human proportions, and everyday life and are fascinated by the meter of time and metaphysical expression of infinity as well as numerical mysticism and the harmonic orders [2].

Since the 1960s, conceptual artists such as Hanne Darboven, On Kawara, and others have explored the significance of numbers as universal language and measurement as well as the interrelationships of numbers and their role within a larger ordering system. Whether simply written out or used as graphic elements, numbers have served as an artistic medium to carry specific messages, to visualize complex systems, and to convey metaphysical meanings.

In this essay, we focus on another aspect of numbers in art, namely, the explicit use of various natural number sequences. In particular, we look at the role of the natural numbers in the work of Roman Opalka, the prime number sequence in the work of Rune Mields, the Fibonacci number sequence often used by Mario Merz, and the U-sequence in the work of Michael Müller. Of interest are not only the wider socio-cultural contexts and larger philosophical meanings behind the use of these sequences by contemporary artists, but also the mathematical theory underlying the respective sequences, which we will explore in detail.

The mathematics of the natural numbers and sequences of natural numbers such as the Fibonacci numbers are studied within the mathematical field of number theory [3]. Even though there exist uncountably many sequences of natural numbers, that is, the set of all sequences cannot be enumerated, those sequences that have mostly been studied in mathematics are identified with a number in Neil Sloane's *The online encyclopedia of integer sequences* [4].[i] The numbering of Sloane's encyclopedia is used in this essay, for example, the sequence of even numbers is referred to as [A005843].

Roman Opalka: natural numbers and the passage of time

The Polish artist Roman Opalka (*1931) uses the sequence of natural numbers [A001477]

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, … .

to visualize the passage of time. In 1965, he began his lifelong task of painting numbers in sequence, beginning with 1, then 2, going on, in theory, to infinity. Accordingly, the name of his work is: *OPALKA 1965/1-∞* (illustration 1). It is an ongoing work, a continuous process that will never be completed and only end when the artist dies. Traditionally, artists have used symbols such as clocks or the human skull to allegorically represent the passing of time and the fleetingness of life and life's vanities. Opalka uses instead the natural numbers to visualize the passing of time. Purposely stripped of all manner of expression, the paintings constitute a record of the artist counting from the number one towards infinity. Each work in the series, entitled *Detail*, is identical in size (1.96 x 1.35 m) and is covered with numbers painted in white, in level rows, from edge to edge. The first painting in the series includes the numbers 1 to 35328; the second panel then begins with 35329.[ii] The artist brightens the originally black canvas by 1% from painting to painting, so that the second canvas is 1% brighter than the first, the third canvas is 1% brighter than the second and 1.99% brighter than the first, the fourth canvas is 2.97% brighter than the first, and so on. The «last» canvas in the series will consist of white numerals on an almost white background, again emphasizing the continuity of his work and the passage of time. In the tradition of Kazimir Malevich's *White on White* painting of 1918, Opalka is continually approaching infinity with his series of numbers and ever-lighter canvases, but will never reach it. The paintings, an ongoing numerical record of the passage of time, are accompanied by a recording of the artist counting aloud in his native language and by photographs of the artist taken at regular intervals as he works on the canvases. Having spent more than half of his life on this single work, Opalka has meanwhile passed the number 5000000; there are now more than 200 panels in the series.

[i] In fact, there are as many sequences of natural numbers as there are numbers on the number line.

[ii] Putting Opalka's first detail under scrutiny, one can find some mistakes in his attempt to write out all natural numbers in increasing order. For example, the numbers 34009, 33010, 33011, 34015, 34016 are listed successively.

Rune Mields: chaos and order in the prime numbers

Prime numbers are positive integers larger than 1 that are only divisible by 1 and by themselves. Numbers which are not prime and which therefore have nontrivial divisors are called composite numbers. The sequence of primes starts with

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, … .

The Greek philosopher and mathematician Euclid (circa 325 – 265 B. C.) showed that the sequence of primes is infinite, just like the sequence of natural numbers [5]. In fact, if there were only finitely many prime numbers, then every natural number would be divisible by at least one of those finitely many prime numbers. Therefore, the number computed by adding 1 to the product of all these finitely many prime numbers will be divisible by at least one of these prime numbers. Since the product itself is also divisible by that same prime number, the difference, namely 1, must be divisible by that prime number as well. This is obviously nonsense and, therefore, the assumption that there are only finitely many primes must have been incorrect in the first place. Hence, there are infinitely many prime numbers.

This classical proof by contradiction helps neither to find infinitely many prime numbers, nor to determine whether a given number is prime. Indeed, it is simple to show that 229 is prime, in fact, it is the 50th prime, but a significant effort is needed to check whether a number with several hundred or even several thousand digits is prime. Even though prime numbers with thousands of digits are not used for counting – some scientists estimate that an integer with 130 digits should be sufficient to represent the number of electrons in the observable universe – large primes are in fact useful and play a commercial role in the encryption of information [6].

To find fast algorithms which determine whether a large number is prime remains an active research topic in number theory. A recent and major theoretical achievement in this direction was the justification that the amount of computations needed to see whether an N-digit number is prime does not grow exponentially with N, but can be bounded by $C \cdot N^{12}$ (C is a large constant that does not depend on the number of digits N) [7].

Today, hundreds of volunteers donate computer resources to the *Great Internet Mersenne Prime Search (GIMPS)* project which coordinates the hunt for larger and larger primes [8]. The *GIMPS* project focuses on the discovery of the so-called Mersenne primes, that is, of primes of the form 2^n-1, where n is prime. Their effort delivers a new *largest known prime* approximately twice a year. The current record is $2^{32582657}$-1. It was discovered on September 6th, 2006, and its 9’808’358 digits barely missed the 10 million digits necessary to collect the prize of $100’000 for cooperative computing from the *Electronic Frontier Foundation* [9]. The inclusion of this currently largest known prime, digit by digit, would require approximately an extra 3’800 pages of this issue of *Arkhaï*. This essay alone would consequently be almost 40 centimeters thick.

The quest to find larger and larger prime numbers has occupied mathematicians and artists alike. The Cologne-based artist Rune Mields (*1935) has concentrated in her work on exploring the largest known prime numbers for decades. In the 1970s, she created a series of works visualizing the primes between 1 and 120000 in the Chinese-Japanese Sanju system. When, in 1980, the then largest prime number 2^{444497}-1, a number with 13’396 digits, was discovered, she used a computer printout of this prime number to identify the digits 0 and 9 within this prime, as well as their frequency. After a new large prime number was discovered by the German mathematician Wilfried Keller in 1984, Mields used a printout of this prime for her work *Die Söhne der Mathematik* (The Sons of Mathematics, 1988) [10].

More generally, Mields focuses on the contrast between chaos and order by exploring old and new ordering systems in their historical and cultural continuity. She has worked with 96 known number systems and sequences, among them stick counting, the Fibonacci series, the Sanju prime numbers, the magical square, and the 6 x 12 number-picture system. In her study of prime numbers, Mields has also focused on visualizing repunit primes – primes whose digital presentation is a string of 1s, the only known repunits have 2 digits (11), 13 digits (1111111111111111111), 19 digits, 317 digits and 1’031 digits – and the so-called twin primes 3,5 ; 5,7 ; 11,13 ; 17,19 ; 29,31 ; … Her interest in twin primes touches upon a famous open problem in number theory, namely the *Twin Prime Conjecture* which states that *there are infinitely many prime numbers* p *with the property that* p+2 *is also a prime number* [11]. This problem was already mentioned by Euclid in his *Elements*. The largest known example for twin primes is the pair

$$100314512544015 \cdot 2^{171960}-1,\ 100314512544015 \cdot 2^{171960}+1,$$

each of which has 51’780 digits. Mields explores these mathematical questions from an artist’s point of view and accompanies mathematicians as they attempt to proceed into uncharted territory of number theory.

Mields also represents prime numbers in various visual forms in an effort to reveal structure and provide insight. In her series *Das Sieb des Eratosthenes III* (The Sieve of Eratosthenes III, 1977, illustration 2), she applies a famous procedure, named after the Alexandrian scholar Eratosthenes of the 3rd century B.C. and still taught in schools today, for finding all prime numbers smaller than a given large number. She summarizes this procedure as follows [12]:

1. *2 is a prime number so put a circle round the 2 on your square.*
2. *Now cross out all the other multiples of 2. If you can spot a pattern you will be able to do this very quickly.*
3. *Now put a circle round the next number after 2 that isn't crossed out, that is 3.*
4. *Now cross out all the other multiples of 3. You will find that some of them are crossed out already. Once again, look for a pattern.*
5. *Now move on to the 5. Circle it and cross out all its other multiples. Keep going… all the circled numbers will be prime!!*

While the hunt for bigger and bigger prime numbers necessitates the development of fast algorithms and fast computers, there are many open questions analogous to the *Twin Prime Conjecture* waiting for a «pencil on paper» proof. The *Riemann Hypothesis* [13], likely the most sought after question in mathematics since Andrew Wiles was able to confirm *Fermat's Last Theorem* in 1995, is one of seven open problems in mathematics whose corroboration carries a prize of $1'000'000 from the *Clay Mathematics Institute* [14]. The *Riemann Hypothesis* discusses the distribution of primes among integers. As was shown above, there are infinitely many prime numbers. Still, one can ask what the proportion of prime numbers to composite numbers is, or, loosely speaking, what is the probability that a randomly chosen natural number is prime. The *Riemann Hypothesis* states a more precise estimate of the proportion of primes to composites than the long established *Prime Number Theorem*, which asserts that there are approximately $n/\log(n)$ prime numbers smaller than any given number n. This implies that large numbers are less likely to be prime numbers than small numbers, something which becomes intuitively clear through, for example, *The Sieve of Eratosthenes*. The *Prime Number Theorem's* accuracy is quite impressive. For example, $n/\log(n)$ estimates 21 prime numbers smaller than 100, 144 smaller than 1000, 1'085 smaller than 10000, and 8'685 smaller than 100000, while the precise numbers are 25 primes smaller than 100, 168 smaller than 1000, 1'229 smaller than 10000, and 9'592 smaller than 100000.

The equally famous *Goldbach Conjecture* from 1742 states *that every even integer larger than two can be written as the sum of two (not necessarily distinct)* primes [15]. It is worth noting that while the definition of prime numbers is based on multiplication, both – the *Twin Prime Conjecture* and the *Goldbach conjecture* – are stated in terms of the additive structure of integers. This leads to the question of whether the interplay of multiplication and addition is the reason that these conjectures, though simple to state, have withstood centuries of attacks by the most brilliant mathematicians.

The most recent landmark achievement in the theory of prime numbers is due to Ben Green and Terence Tao (2004) and concerns the additive structure of prime numbers as well. They proved a classical conjecture that states that the sequence of prime numbers contains arbitrarily long arithmetic progressions [16]. That is, for any large number, say 100, there exists a sequence of 100 prime numbers with the property that the difference of two successive prime numbers is always the same. The proof of Green and Tao is certainly not constructive, that is, it does not provide a method that locates arithmetic progressions in the primes ; the longest arithmetic progression known today has only 23 terms and starts with

$$56211383760397, 100758121856257, 145304859952117, \ldots .$$

It was found by Markus Frind, Paul Jobling and Paul Underwood in 2004 [17]. The result of Green and Tao described above is only one of a number of remarkable results which were achieved by collaborations involving Terence Tao. For this reason, Terence Tao, 31 years old, was awarded a *Fields Medal* last year – equivalent to a *Nobel Prize* for mathematicians.

Mario Merz : Fibonacci numbers as « chiffre for life and art »

Another number sequence that has fascinated scholars and artists alike is the sequence of Fibonacci numbers [A000045], named after the Italian mathematician Leonardo of Pisa, also known as Fibonacci. To define them, one has to fix the numbers 0 and 1 as the first Fibonacci numbers in the sequence and declare that, however many Fibonacci numbers have been constructed, the next Fibonacci number is the sum of the two previously constructed ones, in this case the sum of 0 and 1, that is 1. The next Fibonacci number is again the sum of the two previous ones, now 1 and 1 ; their sum equals 2. This procedure defines the obviously infinite Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \ldots .$$

At first glance, the Fibonacci sequence might appear as mystique as the sequence of primes. The Fibonacci sequence is recursively defined as were the prime numbers, so to determine the 50th Fibonacci number by the rule given above, one needs to know the 48th and the 49th Fibonacci numbers. To compute these, the 46th and 47th Fibonacci numbers are required, and so on.

But contrary to the prime numbers, a simple mathematical analysis shows that the recursive procedure can be circumvented and Fibonacci numbers can be computed directly. In other words, the n^{th} Fibonacci number F_n is given by Binet's formula

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

This implies that the computation of the 50^{th} Fibonacci number does not actually require knowledge of the 49^{th}, the 48^{th}, the 47^{th}, or the 1^{st} Fibonacci number. For instance, when n=50, the formula above gives 12586269025, which is exactly the 50^{th} Fibonacci number.

Above, it was shown that the sequence of prime numbers not only possesses infinitely many terms, but has a rich structure which, even though it has been studied for two thousand years, is still far from being fully understood. Its structure is so rich that simple statements such as the existence of infinitely many twin primes are still awaiting proof. At the same time, considerable computer power is used nowadays to push the boundary of known prime numbers further and further. In contrast, Fibonacci numbers have a very simple, well-understood structure as shown by the formula above. Similar questions to those asked about the prime numbers are easily answered. For example, not every even number is the sum of two Fibonacci numbers (12 is not), and there are not infinitely many twin Fibonacci numbers. In fact, the authors of this paper are not aware of any catchy open conjecture involving the Fibonacci numbers; furthermore, the formula above allows one to easily find a Fibonacci number with 10'000'000, or even more digits.

From a mathematical point of view, the infiniteness of Fibonacci numbers has been resolved and its simplicity is exposed. On the other hand, the relation of the Fibonacci sequence to the golden ratio and their relevance for growth in nature still plays and is expected to continue to play a rich role in the visual arts.

The Italian Arte Povera artist Mario Merz (*1925) came across the Fibonacci number sequence in 1967 and made it the basis of his artistic work. He was particularly attracted by the fact that Fibonacci sequences appear in biological settings and can be found in the growth patterns of leaves, snail shells, pine cones, sundry fruits and vegetables, and the skins of reptiles [18]. The numbers and arrangements of petals, seeds, and plants that are formed in spirals such as pinecones, pineapples, and sunflowers also adhere to the Fibonacci series. Like the Greek philosopher and geometer Pythagoras (circa 570 – 490 B.C.) who saw in numbers the ultimate reality, Merz perceived the Fibonacci numbers as something alive, dynamic and growing, a « chiffre for life and art ».

In 1968, Merz began to work on his famous igloos, revealing the prehistoric and tribal features hidden within the present time and space. These dome-shaped temporary constructions, among them *Pythagoras' Haus* of 1994 (illustration 3), express the artist's preoccupation with the fundamentals of human existence : shelter, food, and the human relationship to nature. Each of these archetypal dwellings is built specifically for the exhibition in which it is shown, the materials – metal tubing, glass, sand bags, branches, or stone – often being indigenous to the location. The organic aesthetics of his installations are simultaneously contrasted and enhanced by the use of neon numbers that refer to the Fibonacci series. Drawing upon this exponentially growing mathematical sequence, Merz emphasizes the growth patterns of natural life in his constructions. His usage of the Fibonacci series as an organizing principle of his works not only suggests an instance of time as subject matter or a meditation on infinity, but also intrinsically links natural patterns, human behavior, and artistic creation.

The golden ratio was considered by the Greeks to be the most aesthetically pleasing visual proportion and has dominated discourse on art from antiquity to modern times. Euclid described the golden ratio in his *Elements*, a collection of thirteen books on geometry and number theory, as follows : *« A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less »* [19]. The division of a line *AB* at any point *M* on the line leads to the two ratios of lengths *AB : AM* and *AM : MB*. But there is only one point *M* for which the ratio is equal, and the ratio is then $\frac{1+\sqrt{5}}{2}$ which is often called Φ (capital phi). Its reciprocal is $\phi = \frac{\sqrt{5}-1}{2}$ (lower case phi). Both of these are variously called the golden number or golden ratio, golden section, golden mean, or the divine proportion.

The golden ratio $\frac{1+\sqrt{5}}{2}$ clearly occurs also in Binet's formula for the Fibonacci sequence. Since $\left(\frac{1-\sqrt{5}}{2}\right)^n$ approaches 0 with increasing n, Binet's formula is actually dominated by the term $\left(\frac{1+\sqrt{5}}{2}\right)^n$. This explains the exponential growth of the Fibonacci sequence, but also implies that the ratio of successive Fibonacci numbers approaches the golden ratio, that is,

$$\frac{F_{n+1}}{F_n} \approx \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n}} = \frac{1+\sqrt{5}}{2}.$$

The golden ratio was believed to evoke emotional or aesthetic feelings and to be particularly pleasing to the human eye [20]. It has been identified in the construction of the great pyramids in Egypt, the *Pyramid of the Sun* at Teotihuacán in Mexico, and in Greek temples such as the Parthenon in Athens [21]. Pythagoras identified the golden ratio in the proportions of the human body; his revelations on the proportions of the human figure had a tremendous impact on Greek and Renaissance art and architecture.

In the early 16th century, the golden ratio was rediscovered by Luca Pacioli (1445–1517). In his treatise *Divina proportione* (On the Divine Proportion, 1509), he attributes to it five divine attributes. In the first four, he stated that it is unique like God, is a single proportion in three terms as the trinity is one substance in three persons, cannot be expressed by a rational number as god cannot be described in words, and is, like God, always similar to itself. In the fifth, Pacioli compares the «divine proportion» to the Platonic quintessence: *«God confers being to the celestial virtue, called by the other name 'fifth essence', and through that one to the other four simple bodies, that is, to the four earthly elements... and so through these to every other thing in nature. Thus this our proportion is the formal being (according to Timaeus) of heaven, attributing to it the figure of the solid called Duodecahedron* [Dodecahedron]*, otherwise known as the solid of twelve pentagons»* [22]. Pacioli's treatise greatly influenced Renaissance artists such as Leonardo da Vinci, Michelangelo, Raphael, and Albrecht Dürer who used the golden ratio in their works. In modern times, the golden ratio can be found in the works of Georges Seurat and Paul Signac as well as Piet Mondrian and the architect Le Corbusier.

Michael Müller: the mystique of Ulam numbers

Like the Fibonacci number sequence, any Ulam number sequence is determined by its first two terms [23]. The most famous choice starts with 1 and 2; the resulting Ulam sequence is then called (1,2)-Ulam sequence or, simply, U-sequence [A002858]. The definition of the remaining terms in an Ulam sequence is again based on a recursive rule. Namely, the next term of any Ulam sequence is given by the next smallest number that can be written in exactly one way as the sum of two previously determined Ulam numbers. Hence, 3=1+2 is an U-number, and so is 4=1+3. The number 5=1+4=2+3 is not a U-number, but 6=2+4 is again a U-number. The U-number sequence therefore begins with

1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, 72, … .

Like the Fibonacci sequence, the U-numbers are defined on the basis of addition only, so it could be expected that U-numbers are not more complex than the Fibonacci numbers. This is not the case as is described below.

The first question considered here is whether there exist infinitely many U-numbers. To confirm this, the opposite is assumed, namely that there are only finitely many U-numbers. In that case, there would be two largest U-numbers. But their sum is a number that could only be written in one particular way as the sum of two distinct U-numbers, that is, as the sum of these two largest ones. This clearly gives rise to another, larger U-number and thereby contradicts the assumption that made it possible to pick the largest two U-numbers earlier. The assumption that there are only finitely many U-numbers must have been wrong, and, therefore, there are infinitely many U-numbers.

The German artist Michael Müller (*1970) has been fascinated by the structure of Ulam numbers since the early 1990s. Müller's interest in the works of the Polish mathematician Stanislaw Ulam (1909 – 1984) originated in Ulam's contributions to war technology as a member of the Manhattan Project at Los Alamos, New Mexico. There, Ulam invented nuclear pulse propulsion and, in collaboration with the Hungarian physicist C. J. Everett, he improved Edward Teller's early model of the hydrogen bomb. Ulam was furthermore an early proponent of using computers to perform mathematical experiments. Looking at Ulam's many contributions to 20th-century science and technology, Müller also came across one of Ulam's contributions to mathematics, the so-called Ulam numbers [24]. Despite his growing interest in the U-sequence, Müller did not become aware of the number theory literature spurred by the creation of the Ulam sequences.

Fascinated by the U-numbers, with their simple definition yet complex structure, he employed software engineers to implement an algorithm for the computation of U-numbers on powerful computers. He attempted to break down the complexity of the Ulam numbers and to increase his understanding of their structure and density within the natural numbers by indulging himself in an overwhelming quantity of computed U-numbers. Like mathematicians before him, he failed to decipher the sequence of U-numbers.

For example, it is known that there are infinitely many U-numbers, but, as with prime numbers, the obvious question about the density of U-numbers relative to the natural numbers remains to be addressed. In fact, Stanislaw Ulam conjectured that as with prime numbers, the probability that a random choice of an integer would result in a U-number gets smaller as the chosen number gets larger. This conjecture has not been proven or disproven, but computations of all U-numbers smaller than $4 \cdot 10^7$, which were carried out by Jud McCranie, seem to imply the opposite: on average, there are apparently 2 U-numbers in every segment of 27 natural numbers [25].

The (1,2)-Ulam sequence described above is still hardly understood. Surprisingly, the complications suddenly vanish when the starting values are changed. For example, the (2,7)-Ulam sequence [A003668] is given by

2, 7, 9, 11,13, 15, 16, 17,
19, 21, 25, 29, 33, 37, 39, 45, 47, 53, 61, 69, 71, 73, 75, 85, 89, 101,
103, 117, 133, 135, 137, 139, 141, 143, 145, 147, 151, 155, 159, 163,
165, 171, 173, 179, …

and the differences of successive terms are

5, 2, 2, 2, 2, 1, 1,
2, 2, 4, 4, 4, 4, 2, 6, 2, 6, 8, 8, 2, 2, 2, 10, 4, 12, 2, 14, 16, 2, 2, 2, 2, 2,
2, 2, 4, 4, 4, 4, 2, 6, 2, 6, 8, 8, 2, 2, 2, 10, 4, 12, 2, 14, 16, 2, 2, 2, 2, 2,
… .

Aside from its first seven terms, the resulting sequence of differences is apparently periodic and repeats itself every 26 numbers [26,27], a phenomenon that is expressed in mathematical terms by saying that the (2,7)-Ulam sequence is regular [27]. Consequently, the density of regular Ulam sequences is easily calculated. The list above shows that there are 33 (2,7)-Ulam numbers smaller than 142, and the regularity implies that there are 59 smaller than 266, 85 smaller than 390, and so on. Therefore, the density is 33/142, 59/266, 85/390, 111/515, 137/639,… and the fractions approach the density 13/62. Therefore, about 21% of the integers are elements of the (2,7)-Ulam sequence. The regularity of this Ulam sequence also implies that there are no (2,7)-Ulam twins aside of 15 and 16, and 16 and 17. Arbitrarily long arithmetic progressions exist as well, for example

19, 145, 271, 397, 523, 649, 775, 901, 1027, 1153, 1279, 1405, 1531, 1657

is an arithmetic progression of length 15 that is contained in the (2,7)-Ulam sequence and that can be continued to obtain arithmetic progressions of any length.

The (2,7)-Ulam sequence has only 2 even terms; in fact, it has been shown that any Ulam sequence with finitely many even terms is regular [28]. But even this rather general theorem does not help to decipher the (1,2)-Ulam sequence. In fact, among the first 200 elements of the (1,2)-Ulam sequence there are 105 even terms.

In his attempts to understand the density of the (1,2)-Ulam sequence, Müller compiled all numbers that are U-numbers and those that are not into separate books, realizing his *Ulam Rot/Grün* (Ulam Red/Green, 2005, illustration 4). Thirteen DIN A4 format books list the numbers from 1 to 226555 in one column per page and one number per line. Whereas a single green volume includes all U-numbers, the twelve red volumes include all other natural numbers; the books represent the inseparability of the natural numbers.

As with the question on the existence of infinitely many twin primes that mathematicians and artists such as Rune Mields have explored, Müller realized that the only successive numbers in his long list of U-numbers are 1,2; 2,3; 3,4; and 47,48. He therefore conjectured that 47 and 48 would be the largest twin pair of U-numbers. In fact, he devoted two art works to this phenomenon and announced a prize to those proving or disproving his assertion. In his *47 48* in 2005 (illustration 5), Müller embossed the numbers 47 and 48 into pink blotting paper and mounted them onto grey cardboard. The subheadings of the two pieces read accordingly; they formulate his assumption that 47 and 48 is the last pair of directly sequential Ulam numbers. Without knowing, the artist had restated a conjecture that was first made by Bernardo Recaman in 1973 and which is therefore almost as old as the U-numbers themselves [29]. The problem is included in the book *Open Problems in Number Theory* [30]. Its renowned author, Richard K. Guy, might have slightly exaggerated the difficulty of solving this problem by stating that he believes that this problem will «never» be solved [31].

Müller realized that lining up U-numbers in increasing order would not help him to further understand them, so he decided to develop new visualization methods that would help to decipher these sequences. His first method was an adaptation of one of Ulam's best known contributions to number theory, namely the Ulam spiral, which Ulam used to obtain new insights into the sequence of prime numbers [32]. As with U-numbers, the line up of primes in increasing order appears as if one has just chosen numbers, that is, odd numbers, at random. The Ulam spiral is obtained by numbering the squares on a sheet of graph paper, starting from the center and proceeding in counter-clockwise spirals. The boxes containing prime numbers are then marked and, surprisingly, the prime numbers appear not to be distributed on the graph paper at random, but pile up on specific diagonals. Müller borrowed this procedure, but marked U-numbers rather than prime numbers on his spiral of natural numbers. As Müller demonstrates in his 2006 *Ulam Spirale* (Ulam Spiral, illustration 6), the resulting picture appears to contain points at random. While noticing the randomness, Müller also saw the Christian cross appear in the middle of his page, adding, in his words «infiniteness of divinity» to the «infiniteness of the unordered» [33].

Müller's 2005 work *Ulam Ringe* (Ulam Rings, illustration 7) presents another mathematically intriguing visualization of the Ulam number sequence. The first circles that are drawn have diameters 1 and 2, corresponding to the first U-numbers. They touch at one point. The next U-number is 3, so a circle of diameter 3 is required. Since the sum, which determines the 3, involves the U-numbers 1 and 2, one chooses a circle of diameter 3, which touches both previously drawn circles again at one point only. The following U-number is 4 and a circle of diameter 4, touching the circles with diameter 1 and 3 at one point each, is drawn. The next circle has diameter 6 and likewise touches circles of diameter 2 and 4, and so on. It is easy to see that there are at most two circles of a given diameter touching two previously drawn circles. However, Müller's experiment leads to the question, whether the structure of the U-numbers guarantees that circles corresponding to U-numbers whose sum gives a new U-number are not too far apart to be touched by a single circle of the prescribed diameter. This is necessary for the process of drawing circles in the described way to be continued indefinitely.

Müller's set of pencil drawings *Ulam Gebirgszug* (Ulam Mountain Range, 2005, illustration 8) represents a possibility to visualize the Ulam sequence: the sequence from 260 to 502 merges into a spatial diagram. The shifted sheets of paper evoke the flow of a river that would culminate, according to the principle of the sequence, in the number 1 to the right and widen towards infinity on the left [34].

As for other artists interested in number sequences, Müller's fascination with Ulam numbers is more metaphysical than mathematical. His art represents his own approach to the study of the Ulam numbers. He did not borrow results from number theory and it is therefore quite remarkable that his search has led him to the same questions as those of the number theorists. For Müller, numbers not only serve as a tool to create structure, but also provide a means to explore infinity. Responding to the human desire to give order to cosmic chaos, he concentrates in his work on expressing infinity in finite space. In his quest to visualize infinity through art, Müller explores the three realms of the human search for infinity: science, space, and religion. He is interested in mathematics in order to draw conclusions about infinity; his works on the U-sequence use the mathematical systems of a specific number series to visualize infinite processes from an artistic-scientific viewpoint. Another problem that has informed Müller's art is the question how to overcome the finiteness of real space, that is, how to deal with the discrepancy between the finiteness of the space of human perception and the presumed infiniteness of universal, cosmic space. His works on star constellations focus on the exploration of cosmic space in astronomy and philosophy. Religious thought is a third aspect, which is only indirectly referred to by Müller in his translations of religious texts into other languages. Thus, Müller's works are results of human imagination at the intersection of art and science, philosophy and religion; his visual expressions of the infiniteness of number series also stand for the infiniteness of the universe and the infiniteness of God.

The fascination with number sequences is shared by Roman Opalka, Rune Mields, Mario Merz, and Michael Müller. Each of these artists chose a specific number sequence and used it as a means to explore and express a particular theme going beyond usual human comprehension. Opalka uses natural numbers to capture the continuous and endless passage of time. Mields explores chaos and order by means of the complex and only partially understood system of prime numbers. Merz focuses on growth in nature, which historically has been associated with the Fibonacci sequence, and Müller attempts to decipher the U-sequence in his quest to grasp the infinity of mathematics, space, and divinity. The interest in mathematics per se distinguishes Mields and Müller from Opalka and Merz. Both, Mields and Müller, have consulted with mathematicians and their art continues to evolve as they are drawn deeper and deeper into the uncharted territory of number theory.

Götz Pfander is with the Mathematics Group, School of Engineering and Science, Jacobs University Bremen, 28759 Bremen, Germany (e-mail: g.pfander@iu-bremen.de).

Isabel Wünsche is with the Art History Group, School of Humanities and Social Sciences, Jacobs University Bremen, 28759 Bremen, Germany (e-mail: i.wunsche@iu-bremen.de).

References

[1] «Zahlen und Zahlenfolgen» (Numbers and Number Sequences), in: Dietmar Guderian, *Mathematik in der Kunst der letzten dreißig Jahre* (Mathematics in the art of the last thirty years), Ebringen Br., 1990, pp. 13–22.
[2] Karin von Maur (ed.), *Magie der Zahl* (The Magic of Numbers), Stuttgart, 1997.
[3] Melvyn B. Nathanson, *Elemetary Methods in Number Theory*, New York, 2000.
[4] See http://www.research.att.com/~njas/sequences.
[5] Euclid, *Elements,* Book IX, Proposition 20, see http:// aleph0.clarku.edu/ ~djoyce/java/elements/toc.html.
[6] Douglas Stinson, *Cryptography: Theory and Practice*, 2nd edition, Boca Raton, 2002.
[7] Manindra Agrawal, Neeraj Kayal and Nitin Saxena, «Primes is in P» in: *Annals of Mathematics* 160 (2004) 2, pp. 781–793. See also http://www.cse.iitk.ac.in/users/manindra/publications.html.
[8] See http://www.mersenne.org.
[9] See http://www.eff.org/awards/coop.php.
[10] Rune Mields, «Vortrag in der Staatlichen Akademie der Bildenden Künste in Stuttgart, Juli 1994» (Lecture at the State Academy of the Fine Arts in Stuttgart, July 1994), in: von Maur, pp. 344–347.
[11] Richard K. Guy, *Unsolved Problems in Number Theory,* 3rd edition, New York, 2004.
[12] Rune Mields as cited in: Ulrich Grevsmühl, «Mathematics and Modern Art: Number Investigations», in: *MT 124* September 1988, p. 35.
[13] See note [11].
[14] See http://www.claymath.org/millennium.
[15] See note [11].
[16] Ben Green and Terence Tao, «The primes contain arbitrarily long arithmetic progressions», see http://front.math.ucdavis.edu/math.NT/0404188.
[17] The Prime Pp., see http://primes.utm.edu/glossary/page.php?sort=ArithmeticSequence.
[18] Jay Kappraff, *Connections: The Geometric Bridge Between Art and Science*, 2nd edition, New York, 2001.
[19] Euclid, *Elements*, Book VI, Proposition 30.
[20] See note [18].
[21] For a critical review on the usage of the golden ratio, see: George Markowsky, «Misconceptions About The Golden Ratio», in: *The College Mathematics Journal*, 23, 1 (1992), pp. 2–19.
[22] Luca Pacioli, *De divina proportione* (1509), see http://milan.milanovic.org/math/english/golden/golden5.html.
[23] Stanislaw Ulam, *Problems in Modern Mathematics*, New York 1964.
[24] See note [23].
[25] Stephen Wolfram, *A New Kind of Science*, Champaign, 2002.
[26] James Schmerl and Eugene Spiegel, «On the regularity of some 1-additive sequences», in: *Journal of Combinatorial Theory*, A12, 1994, pp. 172–175.
[27] Steven Finch, «Patterns in 1-Addititve Sequences», in: *Experimental Mathematics*, 1 (1992) 1, pp. 57–63.
[28] See note [27].
[29] Bernardo Recaman, «Questions on a sequence of Ulam», in: *American Mathematical Monthly*, 80 (1973), pp. 919–920.
[30] See note [11].
[31] Personal conversation with the author.

[32] Myron Stein, Stanislaw Ulam and Mark Wells, «A Visual Display of Some Properties of the Distribution of Primes», in: *American Mathematical Monthly*, 71 (1964), pp. 516–520; Myron Stein and Stanislaw Ulam, «An Observation on the Distribution of Primes», in: *American Mathematical Monthly* 74 (1967), pp. 43–44.
[33] Personal conversation with the artist.
[34] Clemens Krümmel, «A journey to dust: Michael Müller's graphic occupations», in: *Michael Müller. 1 m2 ins Schwarz hinein.* Teil 1, Klagenfurt 2005, p. 96.